Spectra of Digital Waveforms

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Periodic Signal Representation

Periodic signal:

\[ x(t \pm kT) = x(t), \quad k = 1, 2, 3, \ldots \]

Period and fundamental frequency:

\[
T_0 = \frac{1}{f_0} \quad \omega_0 = 2\pi f_0 = \frac{2\pi}{T}
\]

Periodic signal representation:

\[
x(t) = \sum_{n=0}^{\infty} c_n \phi_n(t) = c_0 \phi_0(t) + c_1 \phi_1(t) + c_2 \phi_2(t) + \cdots
\]

\( \phi_n(t) \)  \quad\text{Basis functions (periodic with the same period as } x(t))\)

\( c_n \)  \quad\text{Expansion coefficients}
Linear System Response

If we know the response of a linear system to each of the basis functions

\[ \varphi_i(t) \rightarrow \text{Linear System} \rightarrow y_i(t) \]

then (by superposition) the response of the system to \( x(t) \):

\[
x(t) = \sum_{n=0}^{\infty} c_n \phi_n(t) = c_0 \phi_0(t) + c_1 \phi_1(t) + c_2 \phi_2(t) + \cdots
\]

is

\[
y(t) = \sum_{n=0}^{\infty} c_n y_n(t) = c_0 y_0(t) + c_1 y_1(t) + c_2 y_2(t) + \cdots
\]

\[
x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t), \quad t_1 < t < t_1 + T
\]

\[
a_0 = \frac{1}{T} \int_{t_1}^{t_1+T} x(t)dt \quad a_n = \frac{2}{T} \int_{t_1}^{t_1+T} x(t)\cos n\omega_0 tdt \quad b_n = \frac{2}{T} \int_{t_1}^{t_1+T} x(t)\sin n\omega_0 tdt
\]
Example - Fourier Series Representation

\[ v(t) = \begin{cases} 
A & 0 < t < \frac{T}{2} \\
-A & \frac{T}{2} < t < 0
\end{cases} \]

\[ a_0 = \frac{1}{T} \int_{t_1}^{t_1+T} v(t)dt = 0 \]

\[ a_n = \frac{2}{T} \int_{t_1}^{t_1+T} v(t)\cos n\omega_0 dt = 0 \]

\[ b_n = \frac{2}{T} \int_{t_1}^{t_1+T} v(t)\sin n\omega_0 dt = \frac{2A}{n\pi} \left[1 - \cos(n\pi)\right] = \begin{cases} 
\left(\frac{4A}{\pi}\right)\frac{1}{n} & n \text{ odd} \\
0 & n \text{ even}
\end{cases} \]

\[ v(t) = x(t) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right), \quad t_1 < t < t_1 + T \]

\[ v(t) = x(t) = \frac{4A}{\pi} \left[ \sin 2\pi f_0 t + \frac{1}{3} \sin 2\pi (3f_0) t + \frac{1}{5} \sin 2\pi (5f_0) t + \cdots \right] \]
Example - Fourier Series Representation

\[ v(t) = \frac{4A}{\pi} \left[ \sin 2\pi(f_0)t + \frac{1}{3}\sin 2\pi(3f_0)t + \frac{1}{5}\sin 2\pi(5f_0)t + \cdots \right] \]

**Graphs:**

- For \( n = 1 \)
- For \( n = 3 \)
- For \( n = 7 \)
- For \( n = 19 \)
- For \( n = 101 \)
Exponential Fourier Series

**Trigonometric Representation:**

\[ x(t) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right), \quad t_1 < t < t_1 + T \]

\[ a_0 = \frac{1}{T} \int_{t_1}^{t_1+T} x(t) dt \]
\[ a_n = \frac{2}{T} \int_{t_1}^{t_1+T} x(t) \cos n\omega_0 t dt \]
\[ b_n = \frac{2}{T} \int_{t_1}^{t_1+T} x(t) \sin n\omega_0 t dt \]

\[ e^{jn\omega_0 t} = \cos(n\omega_0 t) + j \sin(n\omega_0 t) \]

\[ \cos(n\omega_0 t) = \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} \]
\[ \sin(n\omega_0 t) = \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j} \]

**Euler’s identity**

**Exponential Representation:**

\[ x(t) = c_0 + \sum_{n=1}^{\infty} 2|c_n| \cos(n\omega_0 t + \angle c_n) \]

\[ c_n = \frac{1}{T} \int_{t_1}^{t_1+T} x(t) e^{-jn\omega_0 t} dt \]
Example - Fourier Series Representation

\[ x(t) = c_0 + \sum_{n=1}^{\infty} 2|c_n| \cos(n\omega_0 t + \angle c_n) \]

\[ c_n = \frac{1}{T} \int_{t_1}^{t_1+T} x(t)e^{-j\omega_0 t} dt \]

Magnitudes of the spectral coefficients

\[ 2|c_n| = 2 \frac{A\tau}{T} \left| \frac{\sin(n\pi\tau/T)}{n\pi\tau/T} \right| \]

Amplitudes of the spectral coefficients lie on an envelope

\[ 2 \frac{A\tau}{T} \left| \frac{\sin \pi ft}{\pi ft} \right|, \quad \frac{n}{T} = f \]

(Paul, 2006)
Trapezoidal Clock or Data Signals

Pulse parameters:
- Amplitude $A$
- Pulse risetime $\tau_r$
- Pulse falltime $\tau_f$
- Pulse width $\tau$

$x(t) = c_0 + \sum_{n=1}^{\infty} 2|c_n| \cos(n\omega_0 t + \angle c_n)$

$c_0 = A \frac{\tau}{T}$, $\tau_r = \tau_f$

$2|c_n| = 2A \frac{\tau}{T} \left| \frac{\sin(n\pi \tau / T)}{n\pi \tau / T} \right| \left| \frac{\sin(n\pi \tau_r / T)}{n\pi \tau_r / T} \right|$, for $n \neq 0$, $\tau_r = \tau_f$

$D = \frac{\tau}{T} = 50\% \Rightarrow c_n = 0$ when $n$ is even, $\tau_r = \tau_f$
Effect 50 % Duty Cycle on Spectral Content

Trapezoidal pulse, 1V, 10 MHz, 5 ns Risetime

49% Duty Cycle

50% Duty Cycle
Spectral Bounds for Trapezoidal Signals

\[ |c_n^+| = 2|c_n| = 2A \frac{\tau}{T} \frac{\sin\left(\frac{n\pi \tau}{T}\right)}{n\pi \tau/T} \frac{\sin\left(\frac{n\pi \tau_r}{T}\right)}{n\pi \tau_r/T}, \quad \text{for } n \neq 0, \quad \tau_r = \tau_f \]

Expansion coefficients for a trapezoidal signal

\[ \text{Envelope} = 2A \frac{\tau}{T} \frac{\sin(\pi \tau f)}{\pi \tau f} \frac{\sin(\pi \tau_r f)}{\pi \tau_r f}, \quad f = \frac{\tau}{T} \]

Continuous envelope of the spectrum

\[ 20 \log_{10}(\text{Envelope}) = 20 \log_{10}\left(2A \frac{\tau}{T}\right) + 20 \log_{10}\left|\frac{\sin(\pi \tau f)}{\pi \tau f}\right| + 20 \log_{10}\left|\frac{\sin(\pi \tau_r f)}{\pi \tau_r f}\right| \]

Bounds on the one-sided magnitude spectrum of a trapezoidal clock signal
Effect of Rise/Falltimes on Spectral Content

Pulses having small rise/fall times will have larger high-frequency spectral content than will pulses having larger rise/fall times.

In order to reduce high-frequency spectrum in order to reduce the emissions of a product, increase the rise/fall times of the clock and/or data pulses.
Effect of the Risetime on Spectral Content

Trapezoidal pulse, 1V, 10 MHz, 50% Duty cycle

20 ns rise time

5 ns rise time
Bandwidth of Digital Waveforms

Above the 2nd break point, \( f = \frac{1}{\pi \tau_r} \)
the amplitudes of the harmonics are attenuated at a rate of -40dB/decade.

\[
x(t) = c_0 + \sum_{n=1}^{\infty} 2|c_n| \cos(n\omega_0 t + \angle c_n)
\]  (1)

Thus, beyond this frequency, adding more terms in the Fourier series contributes little to the RHS of (1).

\[
x(t) \approx c_0 + \sum_{n=1}^{N} 2|c_n| \cos(n\omega_0 t + \angle c_n)
\]

To be conservative we might choose that frequency to be 3 times this second breakpoint:

\[
f_{\max} = 3 \times \frac{1}{\pi \tau_r} \approx \pi \times \frac{1}{\pi \tau_r} = \frac{1}{\tau_r}
\]

Bandwidth of a digital signal

\[f_{\max} = BW = \frac{1}{\tau_r} \text{ Hz}\]
**Signal Reconstruction**

$f_0 = 100\text{MHz}$, 50\% duty cycle, 5-V trapezoidal waveform with $\tau_r = \tau_f = 1\text{ns}$

$$x(t) \cong c_0 + \sum_{n=1}^{N} 2|c_n| \cos(n\omega_0 t + \angle c_n)$$

**Reconstructed using lower 10 harmonics**

100MHz, 200 MHz, …, 1000MHz

**Reconstructed using lower 5 harmonics**

100MHz, 200 MHz, …, 500MHz

(Paul, 2006)
Reducing Signal Amplitude

Reducing the signal amplitude reduces the frequency content over the entire frequency range.
Reducing Signal Amplitude

Trapezoidal pulse, $f = 10$ MHz, 5 ns rise time, 50% duty cycle

\[
A = 2V \\
A = 1V
\]
Reducing $f_0$ and Maintaining Duty Cycle

$$20 \log_{10} \left( \frac{2A\tau}{T} \right) = 20 \log(2AD), \quad D = \frac{\tau}{T}$$

DC starting level of spectral bounds

$$2|c_n| = 2AD \left| \frac{\sin(n\pi D)}{n\pi D} \right| \left| \frac{\sin(n\pi \tau f_0)}{n\pi \tau f_0} \right|, \quad \text{for } n \neq 0, \quad \tau = \tau_r$$

$c_0 = AD$

When the fundamental frequency of the waveform is reduced the period is increased.

If we want to maintain the same duty cycle, the on-time, $\tau$, has to be proportionally increased.

$$D = \frac{\tau}{T} = f_0 \tau \quad \Rightarrow \quad D_{\text{const}} = \left( f_0 \downarrow \right) (\tau \uparrow)$$

Therefore, reducing the fundamental frequency of the pulse train (increasing the period $T$) while maintaining the same % duty cycle does not affect the starting level.
Reducing $f_0$ and Maintaining Duty Cycle

Fundamental frequency of the wave is reduced ($T$ is increased) and the pulse width is increased to retain the same duty cycle $D$.

This results in moving the first breakpoint in the spectral bound to the left (down) in frequency, so that part of the spectral content in the region 0 dB/decade now rolls off at a rate of -20 dB/decade.

Reducing the fundamental frequency (while maintain the duty cycle) reduces the high-frequency spectral content of the waveform, but does not affect the low-frequency content.
Reducing $f_0$ and Maintaining Duty Cycle

Trapezoidal pulse, 1V, 5 ns rise time, 50% duty cycle

$f_0 = 10$ MHz

$f_0 = 5$ MHz
Maintaining $f_0$ and Reducing Duty Cycle

$$20\log_{10}\left(\frac{2A\tau}{T}\right) = 20\log(2AD), \quad D = \frac{\tau}{T}$$

DC starting level of spectral bounds

$$D = \frac{\tau}{T} = f_0\tau \Rightarrow (f_0)_{\text{const}} = \frac{D}{\tau}$$

If we reduce the duty cycle: $D_2 < D_1$, while the fundamental frequency remains the same, we will lower the starting level.

We will also move the first breakpoint out in frequency.

The first breakpoint for the smaller duty cycle $D_2$ will lie on the -20dB/decade segment for the larger duty cycle $D_1$.

Reducing the duty cycle (the pulsewidth) reduces the low-frequency spectral content of the waveform, but does not affect the high-frequency content.
Reducing $f_0$ and Maintaining Duty Cycle

Trapezoidal pulse, 1V, 5 ns rise time, 10 MHz

$D = 50\%$

$D = 20\%$
References